Improved Bounds for Geometric Permutations*

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Abstract

We show that the number of geometric permutations of an arbitrary collection of n pairwise disjoint convex sets in $\mathbb{R}^d$, for $d \geq 3$, is $O(n^{2d-3} \log n)$, improving Wenger's 20 years old bound of $O(n^{2d-2})$.

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1 Introduction

Let $\mathcal{K}$ be a collection of n convex sets in $\mathbb{R}^d$. A line ℓ is a *transversal* of $\mathcal{K}$ if it intersects all the sets in $\mathcal{K}$. If the objects in $\mathcal{K}$ are *pairwise disjoint*, an oriented line transversal meets them in a well-defined order, called a *geometric permutation*. The study of geometric permutations plays a central role in geometric transversal theory; see [8, 20] for compherensive surveys.

Previous work. In 1985, Katchalski et al. [11] initiated the study of the maximum possible number $g_d(n)$ of geometric permutations induced by a set $\mathcal{K}$ of n pairwise disjoint convex objects in $\mathbb{R}^d$. They constructed, for any $n \geq 4$, a family of n pairwise disjoint convex sets in $\mathbb{R}^2$ that admits $2n-2$ geometric permutations. Edelsbrunner and Sharir [5] showed, five years later, that this bound is tight in the worst case, implying that $g_2(n) = 2n-2$. Wenger [19] proved, also in 1990, that $g_d(n) = O(n^{2d-2})$ in any dimension $d \geq 3$. In 1992, Katchalski et al. [12] generalized their lower bound construction and showed that there exist collections of n pairwise disjoint convex sets in $\mathbb{R}^d$, for any $d \geq 3$, which admit $\Omega(n^{d-1})$ geometric permutations. Since then, closing (or even reducing) the fairly large gap between these upper and lower bounds on $g_d(n)$, in any dimension $d \geq 3$, has remained one of the major long standing open problems in geometric transversal theory.

Several partial steps towards this goal were made in the past decade. Most of them deal with geometric permutations of certain restricted families of pairwise disjoint convex bodies in $\mathbb{R}^d$. For example, Smorodinsky et al. [17] derived a tight upper bound of $\Theta(n^{d-1})$ on the number of geometric permutations induced by an arbitrary collection of n pairwise disjoint balls in $\mathbb{R}^d$. Katz and Varadarajan [14] generalized this result to arbitrary collections of n pairwise disjoint *fat* convex bodies. Other recent works [3, 9, 13, 21] show that the maximum possible number of geometric permutations induced by pairwise disjoint *unit* balls (or, more generally, balls of bounded size disparity) is constant in any dimension.

Other studies bound the number of geometric permutations induced by arbitrary collections of pairwise disjoint convex sets, whose realizing transversal lines belong to some restricted subfamily of lines in $\mathbb{R}^d$. For example, Aronov and Smorodinsky [2] derive a tight bound of $\Theta(n^{d-1})$ on the maximum number of geometric permutations realized by lines that pass through a fixed point in $\mathbb{R}^d$. A recent paper [10] by the authors studies line transversals of arbitrary convex polyhedra in $\mathbb{R}^3$ and derives (as a byproduct) an improved upper bound of $O(n^{3+\varepsilon})$, for any $\varepsilon > 0$, on the number of geometric permutations realized by lines which pass through a fixed line in $\mathbb{R}^3$.

The space of line transversals. Lines in $\mathbb{R}^d$ have $2d-2$ degrees of freedom, and are naturally represented in a real projective space (so-called the *Grassmannian manifold*; see [8]). However, for the purpose of combinatorial analysis, we can represent them (with the exclusion of some "negligible" subset which we may ignore) by points in the real Euclidean space $\mathbb{R}^{2d-2}$; see [8] for more details.

Let $\mathcal{K}$ be a collection of n convex sets in $\mathbb{R}^d$, not necessarily pairwise disjoint. The *transversal space* $\mathcal{T}(\mathcal{K})$ of $\mathcal{K}$ is the set in $\mathbb{R}^{2d-2}$ of all (points representing) the transversal lines of $\mathcal{K}$.

If the sets of $\mathcal{K}$ are pairwise disjoint then any two lines in the same connected component of $\mathcal{T}(\mathcal{K})$ induce the same geometric permutation, so the number of geometric permutations is upper bounded by the number of components of $\mathcal{T}(\mathcal{K})$. In two dimensions, the converse property also holds. That is, lines that stab $\mathcal{K}$ in a fixed order form a single connected component of $\mathcal{T}(\mathcal{K})$; see, e.g., [7]. Thus, according to [5], the transversal space $\mathcal{T}(\mathcal{K})$, of any family $\mathcal{K}$ of n pairwise disjoint convex sets in $\mathbb{R}^2$ has at most $2n-2$ connected components.

The situation becomes considerably more complicated already in $\mathbb{R}^3$: There exist collections of

four (pairwise disjoint) convex sets whose transversal space consists of an arbitrarily large number of connected components [7, 10]. This is a simple instance of the phenomenon that the shape of $\mathcal{T}(\mathcal{K})$ depends on the shape of the sets in $\mathcal{K}$, and may grow out of control if we do not impose any restrictions on the sets of $\mathcal{K}$. This might explain (in part) the difficulty of extending the relatively simple analysis of the number of geometric permutations in $\mathbb{R}^2$ to higher dimensions.

In three dimensions, if the sets in $\mathcal{K}$ have *constant description complexity* (i.e., each set can be described as a Boolean combination of a constant number of polynomial equalities and inequalities of constant maximum degree) then one can obtain sharp bounds on the combinatorial complexity of $\mathcal{T}(\mathcal{K})$ (see [15, 20] for a precise definition). Specifically, the analysis of Koltun and Sharir [15] yields an improved bound of $O(n^{3+\varepsilon})$, for any $\varepsilon > 0$, on the combinatorial complexity, and thus also on the number of connected components of $\mathcal{T}(\mathcal{K})$, for collections $\mathcal{K}$ of this kind. (If $\mathcal{K}$ is a collection of n triangles in $\mathbb{R}^3$, an improved bound of $O(n^3 \log n)$ holds, see [1].) Hence, this also serves as an upper bound on the number of geometric permutations induced by any such collection $\mathcal{K}$. Using this approach, and continuing to assume that the sets in $\mathcal{K}$ have constant description complexity, one can strengthen Wenger's bound of $O(n^{2d-2})$ [19] to apply to the combinatorial complexity of $T(\mathcal{K})$, and not just to the number of geometric permutations. The strength (and beauty) of Wenger's analysis is that it yields this bound without making any assumptions whatsoever on the shape of the sets in $\mathcal{K}$ (other than being convex and pairwise disjoint).

Our results. We first show that the number of geometric permutations admitted by *any* collection of n pairwise disjoint convex sets in $\mathbb{R}^3$ is $O(n^3 \log n)$, thus improving Wenger's previous upper bound on $g_3(n)$ roughly by a factor of n. Our approach can be generalized to higher dimensions, and yields an improved upper bound of $O(n^{2d-3} \log n)$ on $g_d(n)$, for any $d \geq 3$. (In three dimensions, our bound is also a slight improvement of the bound $O(n^{3+\varepsilon})$, for any $\varepsilon > 0$, of [15] for the case where the sets in $\mathcal{K}$ have constant description complexity.)

Here is a brief overview of our solution in $\mathbb{R}^3$. Following the approach of Wenger [19], we represent the directions of transversal lines by points on the unit 2-sphere $\mathbb{S}^2$, separate every pair of objects in $\mathcal{K}$ by a plane, and associate with each such plane the great circle on $\mathbb{S}^2$ parallel to it. We then consider the arrangement $\mathcal{A}$ of the resulting $\binom{n}{2}$ great circles on $\mathbb{S}^2$, which consists of $O(n^4)$ 2-faces. The crucial observation made in [19] is that all transversal lines, whose directions belong to the same 2-face of $\mathcal{A}$, stab the sets of $\mathcal{K}$ in the same order (if the face contains such directions at all). Hence, the number of geometric permutations is upper bounded by the total number of 2-faces of $\mathcal{A}$, implying that $g_3(n) = O(n^4)$.

We improve this bound by showing that the actual number of faces which contain at least one direction of a transversal line (so-called *permutation faces*) is only $O(n^3 \log n)$. Moreover, we show that the overall number of edges and vertices on the boundaries of these faces is also at most $O(n^3 \log n)$.

The analysis proceeds in two steps. First, we use a direct geometric analysis to show that the number of vertices whose four incident faces are all permutation faces is $O(n^3)$. We refer to such vertices as *popular vertices*. Informally, we associate with each popular vertex v (with the possible exception of $O(n^3)$ "degenerate" ones) the intersection line λ_v of the two separating planes h, h' that correspond to the two circles incident to v, and show that λ_v stabs exactly $n-4$ sets of $\mathcal{K}$ (all but the sets in the two pairs separated by h and h', respectively). We then apply, within each of the $\binom{n}{2}$ separating planes, the linear bound on the number of geometric permutations in $\mathbb{R}^2$, due to Edelsbrunner and Sharir [5], combined with a simple application of the Clarkson-Shor probabilistic analysis technique [4], and thereby obtain the overall $O(n^3)$ asserted bound on the

number of popular vertices.

We then use this bound to analyze the overall number of vertices incident to permutation faces. This is achieved by a refined (and simplified) variant of the charging scheme of Tagansky [18].

The analysis can be extended to any dimension $d \geq 4$, but its technical details become somewhat more involved.

The paper is organized as follows. We first derive the nearly-cubic upper bound on $g_3(n)$. To this end, we begin in Section 2 by introducing some notations and the infrastructure, and then establish this bound in Section 3. In Section 4, we extend the analysis to any dimension $d \geq 4$.

2 Preliminaries

The setup in $\mathbb{R}^3$. Let $\mathcal{K}$ be a collection of n arbitrary pairwise disjoint convex sets in $\mathbb{R}^3$. We may also assume, without loss of generality, that the elements of $\mathcal{K}$ are *compact*. Indeed, let $g_d(n)$ be the maximum possible number of geometric permutations induced by a collection of n pairwise disjoint *compact* convex sets in $\mathbb{R}^3$. Let $\mathcal{K} = \{K_1, \ldots, K_n\}$ be a collection of n *arbitrary* pairwise disjoint convex sets in $\mathbb{R}^3$, which induces m geometric permutations, realized by m respective lines $\ell_1, \ldots, \ell_m$. For each $1 \leq i \leq n$ and $1 \leq j \leq m$, let p_{ij} denote an arbitrary point in $K_i \cap \ell_j$. For each $1 \leq i \leq n$, let K_i' denote the convex hull of $\{p_{ij} \mid 1 \leq j \leq m\}$, and observe that K_i' is a compact convex subset of K_i. Hence $\mathcal{K}' = \{K_1', \ldots, K_n'\}$ is a collection of n pairwise disjoint *compact* convex sets, which induces (at least) m geometric permutations (realized by the same lines $\ell_1, \ldots, \ell_m$), so $m \leq g_d(n)$.

We use the following setup, introduced by Wenger [19] and briefly mentioned in the introduction, to analyze geometric permutations of $\mathcal{K}$. Enumerate the elements of $\mathcal{K}$ as $K_1, K_2, \ldots, K_n$. For each $1 \leq i < j \leq n$ we fix some plane h_{ij} which strictly separates K_i and K_j. We orient h_{ij} so that K_i lies in the open negative halfspace h_{ij}^- that it bounds, and K_j lies in the open positive halfspace h_{ij}^+. We represent directions of (oriented) lines in $\mathbb{R}^3$ by points on the unit 2-sphere $\mathbb{S}^2$. Without loss of generality we may assume that the planes h_{ij} are in *general position*, meaning that every triple of them intersect at a single point, and no four meet at a common point.

Each separating plane h_{ij} induces a great circle C_{ij} on $\mathbb{S}^2$, formed by the intersection of $\mathbb{S}^2$ with the plane parallel to h_{ij} through the origin. Equivalently, C_{ij} is the locus of the directions of all lines parallel to h_{ij}. C_{ij} partitions $\mathbb{S}^2$ into two open hemispheres C_{ij}^+, C_{ij}^-, so that C_{ij}^+ (resp., C_{ij}^-) consists of the directions of lines which cross h_{ij} from h_{ij}^- to h_{ij}^+ (resp., from h_{ij}^+ to h_{ij}^-). Note that lines whose directions lie in C_{ij} cannot stab both K_i and K_j. Thus, any oriented common transversal line of K_i and K_j intersects K_j after (resp., before) K_i if and only if its direction lies in C_{ij}^+ (resp., C_{ij}^-).

Put $\mathcal{C}(\mathcal{K}) = \{C_{ij} \mid 1 \leq i < j \leq n\}$, and consider the arrangement $\mathcal{A}(\mathcal{K})$ of the $\binom{n}{2}$ great circles of $\mathcal{C}(\mathcal{K})$. The assumption that the planes h_{ij} are in general position is easily seen to imply that the circles in $\mathcal{C}(\mathcal{K})$ are also in general position, in the sense that no pair of them coincide and no three have a common point. Each 2-face f of $\mathcal{A}(\mathcal{K})$ induces a relation $\prec_f$ on $\mathcal{K}$, in which $K_i \prec_f K_j$ (resp., $K_j \prec_f K_i$) if $f \subseteq C_{ij}^+$ (resp., $f \subseteq C_{ij}^-$). Clearly, the direction of each oriented line transversal λ of $\mathcal{K}$ belongs to the unique 2-face f of $\mathcal{A}(\mathcal{K})$ whose relation $\prec_f$ coincides with the order in which λ visits the sets of $\mathcal{K}$ (as noted above, the direction of λ cannot lie on an edge or at a vertex of $\mathcal{A}(\mathcal{K})$). In particular, the number of geometric permutations is bounded by the number of 2-faces of $\mathcal{A}(\mathcal{K})$, which is $O(n^4)$.

This is the way in which Wenger established this upper bound (in three dimensions) 20 years ago [19]. Moreover, this approach can be extended to any dimension $d \geq 3$, and yields the upper bound $O(n^{2d-2})$ on $g_d(n)$; see [19] and Section 4 below. The main weakness of this argument (as follows from the analysis in this paper) is that most faces of $\mathcal{A}(\mathcal{K})$ do not induce a geometric permutation of $\mathcal{K}$. Specifically, for some faces f the relation $\prec_f$ might have cycles, in which case f clearly cannot contain the direction of a transversal of $\mathcal{K}$. But even if $\prec_f$ is acyclic (and thus a total order) there need not exist any line transversal with direction in f.

More definitions. We need a few more notations. We call a 2-face of $\mathcal{A}(\mathcal{K})$ a *permutation face* if there is at least one line transversal of $\mathcal{K}$ whose direction belongs to f. Note, however, that the directions of the line transversals of $\mathcal{K}$ within a fixed permutation face f is only a subset of f, which need not even be connected; see, e.g., a construction in [7] and the introduction.

Each pair of great circles of $\mathcal{C}(\mathcal{K})$ intersect at exactly two antipodal points of $\mathbb{S}^2$. By the general position assumption, all the circles are distinct, and each vertex v of $\mathcal{A}(\mathcal{K})$ is incident to exactly two great circles. Hence, each vertex is incident to exactly four (distinct) faces of $\mathcal{A}(\mathcal{K})$. Assuming that $|\mathcal{K}| \geq 3$, $\mathcal{C}(\mathcal{K})$ contains at least three great circles, so the boundary of each cell of $\mathcal{A}(\mathcal{K})$ contains at least three vertices. This, and the fact that each vertex is incident to four faces, imply that the number of permutation faces in $\mathcal{A}(\mathcal{K})$ is at most proportional to the overall number of their vertices. It is this latter quantity that we proceed to bound.

We say that vertex v in $\mathcal{A}(\mathcal{K})$ is *regular* if the two great circles C_{ij}, $C_{k\ell}$ incident to v are defined by four *distinct* sets of $\mathcal{K}$; otherwise, when only three of the indices i, j, k, ℓ are distinct, we call v a *degenerate* vertex. Clearly, the number of degenerate vertices is $O(n^3)$, so it suffices to bound the number of regular vertices of permutation faces.

In the forthcoming analysis we will use subcollections $\mathcal{K}'$ of $\mathcal{K}$, typically obtained by removing one set, say K_q, from $\mathcal{K}$. Doing so eliminates all separating planes h_{iq}, for $i = 1, \ldots, q-1$, and h_{qi}, for $i = q+1, \ldots, n$. Accordingly, the corresponding circles C_{iq}, C_{qi} are also eliminated from $\mathcal{C}(\mathcal{K}')$, and $\mathcal{A}(\mathcal{K}')$ is constructed only from the remaining circles. In particular, a regular vertex v of $\mathcal{A}(\mathcal{K})$, formed by the intersection of C_{ij} and $C_{k\ell}$, remains a vertex of $\mathcal{A}(\mathcal{K}')$ if and only if $q \neq i, j, k, \ell$. An edge (resp., face) of $\mathcal{A}(\mathcal{K}')$ may contain several edges (resp., faces) of $\mathcal{A}(\mathcal{K})$. Note that if f' is a face of $\mathcal{A}(\mathcal{K}')$ which contains a permutation face f of $\mathcal{A}(\mathcal{K})$ then f' is a permutation face in $\mathcal{A}(\mathcal{K}')$; the permutation that it induces is the permutation of f with K_q removed.

3 The Number of Geometric Permutations in $\mathbb{R}^3$

Our analysis uses the setup of Tagansky [18], somewhat adapted to our context. To make this paper more self-contained, we will spell out many of the details of the technique as it applies in our context.

Popular vertices and edges. We say that an edge e of $\mathcal{A}(\mathcal{K})$ is *popular* if its two incident faces are both permutation faces. We say that a vertex v of $\mathcal{A}(\mathcal{K})$ is *popular* if its four incident faces are all permutation faces. We establish the upper bound $O(n^3)$ on the number of popular vertices, using a direct geometric argument. The analysis then proceeds by applying two charging schemes. The first scheme results in a recurrence which expresses the number of popular edges in terms of the number of popular vertices. The second scheme leads to a recurrence which expresses the number of vertices of permutation faces in terms of the number of popular edges. The solutions of both recurrences are nearly cubic. Naive (and simpler) implementation of both schemes incurs an extra

logarithmic factor in each recurrence, resulting in the overall bound $g_3(n) = O(n^3 \log^2 n)$. With a more careful analysis of the second scheme, we are able to eliminate one of these factors, and thus obtain the bound $g_3(n) = O(n^3 \log n)$.

3.1 The number of popular vertices

For a regular vertex v of $\mathcal{A}(\mathcal{K})$, formed by the intersection of $C_{ij}, C_{k\ell} \in \mathcal{C}(\mathcal{K})$, we denote by $\mathcal{K}_v$ the collection $\{K_i, K_j, K_k, K_\ell\}$ of the four sets defining (the circles meeting at) v.

Lemma 3.1. *Let v be a regular popular vertex of $\mathcal{A}(\mathcal{K})$, incident to $C_{ij}, C_{k\ell} \in \mathcal{C}(\mathcal{K})$.*
(i) Each pair of sets $K_a \in \mathcal{K}_v$ and $K_b \in \mathcal{K} \setminus \mathcal{K}_v$ appear in the same order in all four permutations induced by the faces incident to v.
(ii) The elements of each pair K_i, K_j and K_k, K_ℓ are consecutive in all four permutations induced by the faces incident to v.

Proof. Any two distinct faces f, g incident to v are separated only by one or two great circles from $\{C_{ij}, C_{k\ell}\}$, so the orders $\prec_f$ and $\prec_g$ may disagree only over the pairs (K_i, K_j) and (K_k, K_ℓ). As a matter of fact, the four permutations are obtained from each other only by swapping K_i and K_j and/or swapping K_k and K_ℓ. This is easily seen to imply both parts of the lemma. □

Lemma 3.2. *Let v be a regular popular vertex in $\mathcal{A}(\mathcal{K})$, incident to $C_{ij}, C_{k\ell} \in \mathcal{C}(\mathcal{K})$. Then the line $\lambda_v = h_{ij} \cap h_{k\ell}$ stabs all the $n-4$ sets in $\mathcal{K} \setminus \mathcal{K}_v$, and misses all four sets in $\mathcal{K}_v$.*

Proof. By definition, λ_v misses every set $K \in \mathcal{K}_v$, because it is contained in a plane separating K from another set in $\mathcal{K}_v$. Hence, it suffices to show that λ_v is a transversal of $\mathcal{K} \setminus \mathcal{K}_v$.

To show this, we fix a set $K_a \in \mathcal{K} \setminus \mathcal{K}_v$ and show that each of the four dihedral wedges determined by h_{ij} and $h_{k\ell}$ meets K_a. The convexity of K_a then implies that λ_v intersects K_a; see Figure 1 (left).

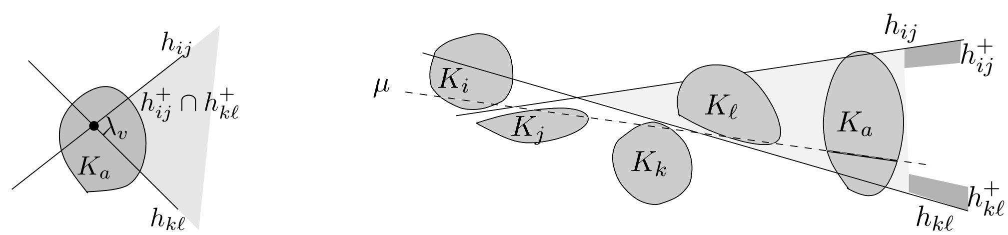

Figure 1: Left: K_a must cross $\lambda_v = h_{ij} \cap h_{k\ell}$ since it meets each of the four incident wedges (one of which is highlighted). Right: The transversal line μ crosses K_a after K_i, K_j, K_k, K_ℓ, so the segment $K_a \cap \mu$ (highlighted) is contained in $h_{ij}^+ \cap h_{k\ell}^+$.

Lemma 3.1 implies that K_a lies at the same position in each of the four permutations induced by the faces incident to v. Without loss of generality, assume that the consecutive pair K_i, K_j appears in these permutations before the consecutive pair K_k, K_ℓ. Then either K_a precedes both pairs in all four permutations, or appears in between them, or succeeds both of them. In what follows we assume that K_a succeeds both pairs in all the permutations, but similar arguments handle the other two cases too.

Consider the permutation π_1 induced by the face f_1 incident to v and lying in $C_{ij}^+ \cap C_{k\ell}^+$, and let μ be a line transversal which induces π_1. Since the direction of μ lies in $C_{ij}^+ \cap C_{k\ell}^+$, it follows that μ crosses h_{ij} from the side containing K_i to the side containing K_j, and it crosses $h_{k\ell}$ from

the side containing K_k to the side containing K_ℓ. Hence K_i precedes K_j and K_k precedes K_ℓ in π_1. Moreover, μ crosses h_{ij} in between its intersections with K_i and K_j, and it crosses $h_{k\ell}$ in between its intersections with K_k and K_ℓ. Thus, $\mu \cap K_a$ lies in $h_{ij}^+ \cap h_{k\ell}^+$; see Figure 1 (right). That is, K_a intersects the dihedral wedge $h_{ij}^+ \cap h_{k\ell}^+$. Fully symmetric arguments, applied to the permutations induced by the three other faces f_2, f_3, f_4 incident to v, show that K_a intersects each of the three other dihedral wedges determined by h_{ij} and $h_{k\ell}$, which, as argued above, implies that λ_v stabs K_a. As promised, slightly modified variants of this argument (with different correspondences between the wedges around λ_v and the faces around v) handle the cases where K_a precedes both pairs K_i, K_j and K_k, K_ℓ in all four permutations, or appears in between these pairs. $\square$

Theorem 3.3. *Let $\mathcal{K}$ be a collection of n pairwise disjoint compact convex sets in $\mathbb{R}^3$. Then the number of popular vertices in $\mathcal{A}(\mathcal{K})$ is $O(n^3)$.*

Proof. Note first that each popular vertex must be regular. Indeed, if v is a degenerate popular vertex incident to, say, $C_{ij}, C_{ik} \in \mathcal{C}(\mathcal{K})$, then, arguing as in Lemma 3.1, each of the two pairs K_i, K_j and K_i, K_k appears consecutively in each of the four permutations near v. Let f be one of the four permutation faces incident to v, and assume, without loss of generality, that $K_k \prec_f K_i \prec_f K_j$. Let g be the permutation face neighboring to f and separated from it only by the circle C_{ik}. Then we must have $K_i \prec_g K_k \prec_g K_j$, contradicting the fact that K_i, K_j are consecutive also under $\prec_g$.

Now let v be a regular popular vertex in $\mathcal{A}(\mathcal{K})$, incident to $C_{ij}, C_{k\ell} \in \mathcal{C}(\mathcal{K})$, and let $\lambda_v = h_{ij} \cap h_{k\ell}$ be the line considered in Lemma 3.2. Put $K_q^* = K_q \cap h_{ij}$, for each index $q \neq i, j$, and denote by $\mathcal{K}^*$ the collection of these $n-2$ planar cross-sections within h_{ij}. Clearly, all sets in $\mathcal{K}^*$ are pairwise disjoint, compact, and convex.

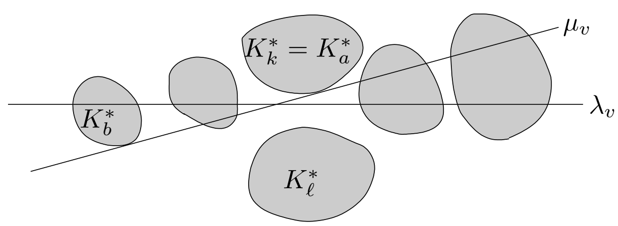

Figure 2: View inside h_{ij}: The line $\lambda_v = h_{ij} \cap h_{k\ell}$ misses K_k^*, K_ℓ^* but stabs all other sets in $\mathcal{K}^*$. The line μ_v is tangent to $K_a^* = K_k^*$ and to K_b^*, so it misses only K_ℓ^*.

By Lemma 3.2, λ_v lies in h_{ij}, stabs all the sets in $\mathcal{K}^* \setminus \{K_k^*, K_\ell^*\}$ (so they are all nonempty) and misses the two sets K_k^*, K_ℓ^*. (As can be easily verified, both of K_k^*, K_ℓ^* are also nonempty, although our analysis does not rely on this property.)

Translate λ_v within h_{ij} until it becomes tangent to some set $K_a^* \in \mathcal{K}^*$, and then rotate the resulting line around K_a^*, say counterclockwise, keeping it tangent to that set, until it becomes tangent to another set $K_b^* \in \mathcal{K}^* \setminus \{K_a^*\}$. The sets $K_k^*, K_\ell^*, K_a^*, K_b^*$ need not all be distinct, so the resulting extremal tangent μ_v misses *at most* two sets of $\mathcal{K}^*$ and intersects all the other sets; see Figure 2.

We charge λ_v to μ_v, and argue that each extremal line μ in h_{ij}, which is tangent to two sets of $\mathcal{K}^*$ and misses at most two other sets of $\mathcal{K}^*$, is charged in this manner at most twice. Indeed, by the general position assumption, μ lies in a single plane h_{ij}. Within that plane, if μ misses two sets of $\mathcal{K}^*$ then these must be the sets K_k^*, K_ℓ^*. If μ misses only one set of $\mathcal{K}^*$ then this set must be one of the sets K_k^*, K_ℓ^*, and the other set is one of the two sets μ is tangent to. Finally, if μ does

not miss any set of $\mathcal{K}^*$ then K_k^*, K_ℓ^* are the two sets μ is tangent to. Hence μ determines at most two quadruples K_i, K_j, K_k, K_ℓ whose lines λ_v can charge μ, and the claim follows.

It therefore suffices to bound the number of extremal lines μ charged in this manner. This can be done using the Clarkson-Shor technique [4], by observing that each such line μ is defined by two sets of $\mathcal{K}^*$ (those it is tangent to; any such pair of sets determine four common tangents) and is "in conflict" with at most two other sets (those that it misses). Thus, the Clarkson-Shor technique implies that the number of lines μ_v is $O\left(L_0(n/2)\right)$, where $L_0(r)$ is the (expected) number of extremal lines which are transversals to a (random) sample of r sets of $\mathcal{K}^*$. Edelsbrunner and Sharir [5] establish an upper bound of $O(r)$ on the complexity of the space of line transversals to a collection of r pairwise-disjoint compact convex sets in the plane, implying that $L_0(r) = O(r)$. Hence the number of charged lines μ in a single plane h_{ij} is $O(n)$, for a total of $O\left(\binom{n}{2} \cdot n\right) = O(n^3)$. Since, as noted above, each line is charged at most twice in its plane, this also bounds the number of popular vertices. $\square$

3.2 The number of popular edges

We next bound the number of popular edges in $\mathcal{A}(\mathcal{K})$, using the bound on popular vertices just derived. We define an *edge border* in $\mathcal{A}(\mathcal{K})$ to be a pair (v, Q), where v is a vertex of $\mathcal{A}(\mathcal{K})$, incident to two great circles $C_{ij}, C_{k\ell}$, and Q is one of the four open hemispheres $C_{ij}^-, C_{ij}^+, C_{k\ell}^-, C_{k\ell}^+$ determined by one of these circles. See Figure 3 (left). Note that Q determines a unique edge e of $\mathcal{A}(\mathcal{K})$ which is incident to v and is contained in Q. If, in addition, e is a popular edge, we say that (v, Q) is a *popular edge border*. For the purpose of the analysis, we will also refer to (v, Q) as a 0*-level edge border*.

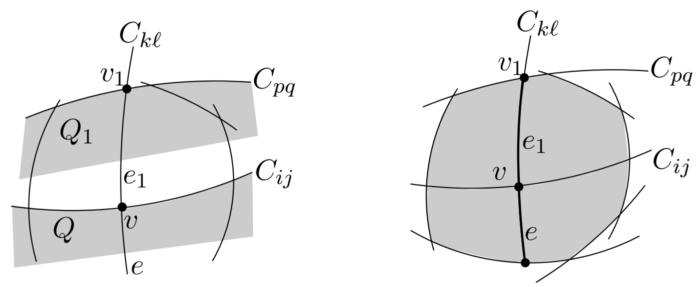

Figure 3: Left: Charging a 0-level edge border (v, Q) to a 1-level edge border (v_1, Q_1). Right: If the edges e, e_1 are both popular then v is a popular vertex.

One useful feature of the border notation is that if (v, Q) is an edge border in $\mathcal{A}(\mathcal{K})$ and $\mathcal{K}'$ is a subcollection of $\mathcal{K}$ so that v is still a vertex of $\mathcal{A}(\mathcal{K}')$, then (v, Q) is also an edge border in $\mathcal{A}(\mathcal{K}')$. The edge e' of $\mathcal{A}(\mathcal{K}')$ associated with (v, Q) in $\mathcal{A}(\mathcal{K}')$ either is equal to e, or strictly contains e (in the latter case both e and e' have v as a common endpoint).

If an edge border (v, Q), which is not a 0-level edge border, becomes a 0-level edge border after removing from $\mathcal{K}$ some single set $K_a \in \mathcal{K}$, we call it a 1*-level edge border*. In this case we say that (v, Q) is *in conflict* with K_a. Note that the set K_a, whose removal makes (v, Q) a 0-level edge border, need not be unique; see Section 3.3 for further discussion.

Clearly, to bound the number of popular edges it suffices to bound the number of 0-level edge borders, which is twice the number of popular edges in $\mathcal{A}(\mathcal{K})$ (each edge is counted once at each of its endpoints).

Since each vertex of $\mathcal{A}(\mathcal{K})$ participates in exactly four edge borders, the number of edge borders which are incident to a degenerate vertex is $O(n^3)$. We bound the number of remaining 0-level edge borders using the following charging scheme.

Let (v, Q) be a 0-level edge border, where v is incident to C_{ij} and $C_{k\ell}$, so that $Q = C_{ij}^+$, say. Let e be the popular edge associated with (v, Q). Trace $C_{k\ell}$ from v away from e (into C_{ij}^-), and let v_1 be the next encountered vertex. Let e_1 be the edge connecting v and v_1. Let C_{pq} be the other circle incident to v_1 and assume, without loss of generality, that v lies in C_{pq}^+. See Figure 3 (left). Note that, assuming $|\mathcal{K}| \geq 3$, we have $C_{pq} \neq C_{ij}$ (i.e., v_1 is not antipodal to v), because otherwise C_{ij} would have intersected only $C_{k\ell}$. One of the following cases must arise:

(i) v_1 is degenerate.

(ii) The edge e_1 is also popular, so v is a popular vertex; see Figure 3 (right).

(iii) e_1 is not popular. Since $C_{pq} \neq C_{ij}$, one of i, j, say i, is different from both p and q. This (and the fact that $i \neq k, \ell$) implies that removing K_i from $\mathcal{K}$ also removes C_{ij} from $\mathcal{A}$, keeps v_1 intact, and makes the appropriate extension of e reach (and terminate at) v_1, thereby making (v_1, Q_1) a 0-level edge border in $\mathcal{A}(\mathcal{K} \setminus \{K_i\})$, where $Q_1 = C_{pq}^+$. See Figure 3 (left).

In case (i) we charge (v, Q) to v_1. The number of degenerate vertices is $O(n^3)$ and each of them can be charged only $O(1)$ times in this manner. Hence, the number of 0-level edge borders that fall into this subcase is $O(n^3)$.

In case (ii) we can charge (v, Q) to v. Since a popular vertex participates in exactly four 0-level edge borders, the number of 0-level edge borders that fall into this subcase is $O(n^3)$, by Theorem 3.3.

In case (iii) we charge (v, Q) to the 1-level edge border (v_1, Q_1). Note that (v_1, Q_1) is charged in this manner only by (v, Q).

Let us denote by $E_0(\mathcal{K})$ (resp., $E_1(\mathcal{K})$) the number of 0-level edge borders (resp., 1-level edge borders) in $\mathcal{A}(\mathcal{K})$. Then we have the following recurrence:

$$E_0(\mathcal{K}) \leq E_1(\mathcal{K}) + O(n^3). \tag{1}$$

To solve this recurrence, we apply the technique of Tagansky [18]. Specifically, we remove from $\mathcal{K}$ a randomly chosen set $K \in \mathcal{K}$, and denote by $\mathcal{R}$ the collection of the $n-1$ remaining sets. A 0-level edge border (v, Q) in $\mathcal{A}(\mathcal{K})$, where v is an intersection point of C_{ij} and $C_{k\ell}$ and is regular, appears as a 0-level edge border in $\mathcal{A}(\mathcal{R})$ if and only if K is different from each of the four sets K_i, K_j, K_k, K_ℓ defining v, which happens with probability $\frac{n-4}{n}$. A 1-level edge border (v, Q) in $\mathcal{A}(\mathcal{K})$ becomes a 0-level edge border in $\mathcal{A}(\mathcal{R})$ if and only if K is in conflict with (v, Q), which happens with probability *at least* $\frac{1}{n}$. No other edge border in $\mathcal{A}(\mathcal{K})$ can appear as a 0-level edge border in $\mathcal{A}(\mathcal{R})$. Hence, we obtain

$$\mathbf{E}\left\{E_0(\mathcal{R})\right\} \geq \frac{n-4}{n} E_0(\mathcal{K}) + \frac{1}{n} E_1(\mathcal{K}), \tag{2}$$

where $\mathbf{E}$ denotes expectation with respect to the random sample $\mathcal{R}$, as constructed above. Combining (1) and (2) yields

$$\frac{1}{n} E_0(\mathcal{K}) \leq \frac{1}{n} E_1(\mathcal{K}) + O(n^2) \leq \mathbf{E}\left\{E_0(\mathcal{R})\right\} - \frac{n-4}{n} E_0(\mathcal{K}) + O(n^2).$$

Denoting by $E_0(n)$ the maximum number of 0-level edge borders in $\mathcal{A}(\mathcal{K})$, for any collection $\mathcal{K}$ of size n with the assumed properties, we get the recurrence

$$\frac{n-3}{n} E_0(n) \le E_0(n-1) + O(n^2),$$

whose solution is easily seen to be $E_0(n) = O(n^3 \log n)$ (see, e.g., [18, Proposition 3.1]).

3.3 The number of permutation faces

Finally, we bound the number of vertices of permutation faces using the bound on popular edges just derived. This will also serve as an upper bound on the number of permutation faces, and thus also on $g_3(n)$. We present the analysis in two stages. The first stage derives the slightly weaker upper bound $O(n^3 \log^2 n)$, but is considerably simpler. The second stage involves a more careful examination of the possible charging scenarios, and leads to a sharper recurrence, whose soution is only $O(n^3 \log n)$.

Each vertex v is incident to exactly four faces of $\mathcal{A}(\mathcal{K})$, so we need to count v with multiplicity of at most 4—once for each permutation face incident to v. For this we extend the notion of borders as follows. The two great circles passing through v partition $\mathbb{S}^2$ into four wedges, or rather slices. Each such slice R contains a unique face f incident to v, and defines, together with v, a *border* (v, R). We call f the face *associated* with (v, R). Similarly to the notation involving edge borders in Section 3.2, we call (v, R) a *popular border*, or a 0-*level border*, if the face associated with (v, R) is a permutation face. It thus suffices to bound the number of 0-level borders in $\mathcal{A}(\mathcal{K})$.

If (v, R) is a border in $\mathcal{A}(\mathcal{K})$ with an associated face f, and $\mathcal{K}'$ is a subcollection of $\mathcal{K}$, so that v is still a vertex of $\mathcal{A}(\mathcal{K}')$, then (v, R) is also a border in $\mathcal{A}(\mathcal{K}')$, except that the face f' of $\mathcal{A}(\mathcal{K}')$ associated with (v, R) may be different from f (or, more precisely, properly contain f).

If a border (v, R), which is not a 0-level border in $\mathcal{A}(\mathcal{K})$, becomes a 0-level border after removing from $\mathcal{K}$ some set K, we call it a 1-*level border*. The set K is said to be *in conflict* with (v, R). Note that K cannot be one of the (at most) four sets defining v, and that a 1-level border may be in conflict with more than one set of $\mathcal{K}$. See Figure 4 (left).

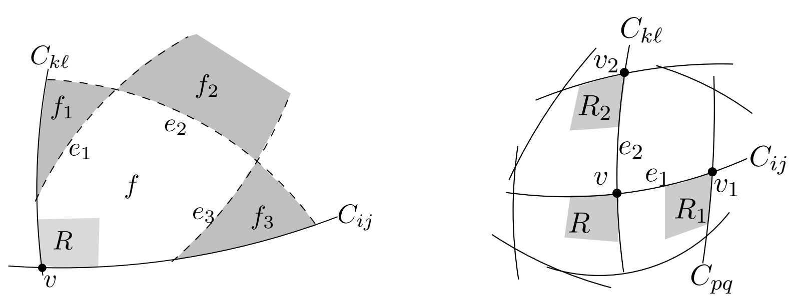

Figure 4: Left: A non-permutation face f, associated with the 1-level border (v, R), is separated from permutation faces f_1, f_2, f_3 by the respective edges $e_1 \subset C_{p_1 q_1}, e_2 \subset C_{p_2 q_2}, e_3 \subset C_{p_3 q_3}$. If none of $p_1, q_1, p_2, q_2, p_3, q_3$ belongs to $\{i, j, k, \ell\}$ then (v, R) is a 1-level border in conflict with each of $K_{p_1}, K_{q_1}, K_{p_2}, K_{q_2}, K_{p_3}, K_{q_3}$. Right: Charging a 0-level border (v, R) to the two 1-level borders (v_1, R_1), (v_2, R_2), along the two edges e_1, e_2 emanating from v away from R.

We bound the number of 0-level borders using a charging scheme similar to that in Section 3.2. Let (v, R) be a 0-level border, and let f be the permutation face associated with it. Note that the

number of borders incident to degenerate vertices is $O(n^3)$. We may therefore assume that v is regular, and let C_{ij} and $C_{k\ell}$ denote the two great circles incident to v (so i, j, k, ℓ are all distinct). Without loss of generality, assume that $R = C_{ij}^+ \cap C_{k\ell}^+$.

Let e_1 and e_2 be the two edges incident to v and emanating from it away from R, where $e_1 \subset C_{ij} \cap C_{k\ell}^-$ and $e_2 \subset C_{k\ell} \cap C_{ij}^-$; see Figure 4 (right). Let v_1 (resp., v_2) be the other endpoint of e_1 (resp., of e_2).

Our charging scheme is based on the following case analysis:

(i) If one of the two edges incident to v and bounding R is popular, we charge (v, R) to this edge. Since the number of popular edges is $O(n^3 \log n)$, and each of them is charged by at most four 0-level borders (twice for each of its endpoints), the number of 0-level borders that fall into this subcase is also $O(n^3 \log n)$.

(ii) If no edge incident to v and bounding R is popular, we charge (v, R) to two 1-level borders, one incident to v_1 and one to v_2. Specifically, consider v_1, say, and let C_{pq} be the circle whose intersection with C_{ij} forms v_1, and assume, again without loss of generality, that v lies in C_{pq}^+. We then charge (v, R) to (v_1, R_1), where $R_1 = C_{pq}^+ \cap C_{ij}^+$. Let f_1 be the face of $\mathcal{A}(\mathcal{K})$ associated with (v_1, R_1) (this is the face whose boundary we trace from v to v_1 along e_1, and it is also incident to v). Since the edge incident to f, f_1 (and to v) is not popular, f_1 is not a permutation face. Clearly, one of the indices k, ℓ, say k, is different from both p, q. Thus, removing K_k keeps v_1 as a vertex in the new spherical arrangement, and makes $C_{k\ell}$ disappear, so both faces f, f_1 fuse into a single larger permutation face contained in R_1. Hence, (v_1, R_1) is a 1-level border which is in conflict with K_k. A fully symmmetic argument applies to v_2. We say that the 1-level borders (v_1, R_1) and (v_2, R_2), which we charge, are the *neighbors* of (v, R) in $\mathcal{A}(\mathcal{K})$.

Note that each 1-level border (v', R') is charged by at most two 0-level borders in this manner (at most once along each of the two edges incident to v' and bounding the face associated with the border).

Let $V_0(\mathcal{K})$ and $V_1(\mathcal{K})$ denote, respectively, the number of 0-level borders and the number of 1-level borders in $\mathcal{A}(\mathcal{K})$ (where we also include degenerate vertices in both counts). Then we have the following recurrence:

$$V_0(\mathcal{K}) \le V_1(\mathcal{K}) + O(n^3 \log n). \tag{3}$$

Indeed, each 0-level border which falls into case (ii) charges two 1-level borders, and each 1-level border is charged at most twice. The number of all other 0-level borders is $O(n^3 \log n)$, as argued above. Combining this inequality with the random sampling technique of Tagansky [18], as in Section 3.2, results in the recurrence

$$\frac{n-3}{n} V_0(n) \le V_0(n-1) + O(n^2 \log n),$$

where $V_0(n)$ is the maximum value of $V_0(\mathcal{K})$, over all collections $\mathcal{K}$ of n pairwise disjoint compact convex sets in $\mathbb{R}^3$. The solution of this recurrence is $V_0(n) = O(n^3 \log^2 n)$, which yields the same upper bound on the number of geometric permutations induced by $\mathcal{K}$.

An improved bound. We next improve the bound by replacing the recurrence (3) by a refined recurrence. Let (v, R) be a 1-level border which is in conflict with $w \ge 1$ sets of $\mathcal{K}$. Then (v, R) becomes a 0-level border in $\mathcal{A}(K \setminus \{K\})$, after removing a random set $K \in \mathcal{K}$, with probability exactly $\frac{w}{n}$. Namely, this happens if and only if K is one of the w sets in conflict with (v, R). We refer to w as the *weight* of (v, R).

In the refined setting, $V_1(\mathcal{K})$ counts the total weight of all the 1-level borders in $\mathcal{A}(\mathcal{K})$, so now the contribution of each 1-level border to $V_1(\mathcal{K})$ is equal to its weight. By an appropriate adaptation of the argument in Section 3.2, we obtain the following *equality*:

$$\mathbf{E}\{V_0(\mathcal{R})\} = \frac{n-4}{n} V_0(\mathcal{K}) + \frac{1}{n} V_1(\mathcal{K}), \tag{4}$$

where $\mathcal{R}$ denotes a random sample of $n-1$ sets of $\mathcal{K}$. This follows by noting that the probability of a 1-level border of weight w to be counted in $V_0(\mathcal{R})$ is $\frac{w}{n}$, and it contributes w to $V_1(\mathcal{K})$.

In the refined charging scheme, each 1-level border (v, R) of weight $w \geq 1$ gets a supply of w units of charge, which it can give to its charging neighboring 0-level borders. Hence, as long as the number of these charging 0-level borders, which is at most two, does not exceed w, (v, R) can pay each of its neighbors 1 unit. Hence, the only problematic case is when $w = 1$ and (v, R) is charged twice. The following technical lemma takes care of this case.

Lemma 3.4. *The number of* 1*-level borders having weight* 1 *and charged by two* 0*-level borders is* $O(n^3 \log n)$.

Before proving Lemma 3.4, we show how to use it to replace 3 by a better recurrence, and thereby establish an improved bound on the number of geometric permutations in $\mathbb{R}^3$.

If a 1-level border (v, R) has only one neighboring 0-level border (v', R') then (v', R') can receive one unit of charge from (v, R), regardless of what the weight of (v, R) is. Similarly, if (v, R) has weight at least 2, and it has two neighboring 0-level borders, each of these 0-level borders can receive one unit of charge from (v, R). The number of remaining 1-level borders, namely the 1-level borders of weight 1 with two neighboring 0-level borders, is $O(n^3 \log n)$, by Lemma 3.4.

To recap, each 0-level border, except possibly for $O(n^3 \log n)$ ones, receives 1 unit of charge from each of its two neighboring 1-level borders. Moreover, the number of charges made to each of the remaining 1-level borders, by its neighboring 0-level borders, does not exceed its weight. Thus, we can replace (3) by the following inequality:

$$2V_0(\mathcal{K}) \leq V_1(\mathcal{K}) + O(n^3 \log n).$$

Combining this with (4) we get

$$\frac{2}{n} V_0(\mathcal{K}) \leq \frac{1}{n} V_1(\mathcal{K}) + O(n^2 \log n) \leq \mathbf{E}\left\{V_0(\mathcal{R})\right\} - \frac{n-4}{n} V_0(\mathcal{K}) + O(n^2 \log n),$$

or

$$\frac{n-2}{n} V_0(\mathcal{K}) \leq \mathbf{E}\left\{V_0(\mathcal{R})\right\} + O(n^2 \log n).$$

Replacing $V_0(\mathcal{K})$, $V_0(\mathcal{R})$ by their respective maximum values $V_0(n)$, $V_0(n-1)$, we thus obtain the recurrence

$$\frac{n-2}{n} V_0(n) \leq V_0(n-1) + O(n^2 \log n),$$

whose solution is easily seen to be $V_0(n) = O(n^3 \log n)$ (again, see [18, Proposition 3.1]).

As mentioned earlier, $V_0(n)$ serves as an upper bound on the number of geometric permutations induced by $\mathcal{K}$. We thus conclude with the following main result of this section.

Theorem 3.5. *Any collection* $\mathcal{K}$ *of* n *pairwise disjoint convex sets in* $\mathbb{R}^3$ *admits at most* $O(n^3 \log n)$ *geometric permutations.*

Proof of Lemma 3.4. Consider a 1-level border (v, R) of weight 1, where v is incident to two great circles C_{ij}, $C_{k\ell}$, which is charged twice. We may assume that v is regular (i.e., the four indices i, j, k, ℓ are distinct), since the number of remaining borders is $O(n^3)$. Let (v_1, R_1) be the 0-level border that charges (v, R) along C_{ij}, and let (v_2, R_2) be the 0-level border that charges (v, R) along $C_{k\ell}$. By construction, both v_1 and v_2 are regular (otherwise they do not charge v). Let $C_{p_1q_1}$ denote the other circle incident to v_1, and let $C_{p_2q_2}$ denote the other circle incident to v_2. Clearly, each index in $\{p_1, q_1, p_2, q_2\}$ which does not belong to $\{i, j, k, \ell\}$ contributes to the weight of (v, R), so, by assumption, there is only one such index, call it q. Since v_1 is regular, neither p_1 nor q_1 belongs to $\{i, j\}$, so (exactly) one of them must belong to $\{k, \ell\}$, say $p_1 = k$ and then $q_1 = q$. Symmetrically, we may assume that $p_2 = i$, say, and then $q_2 = q$. Since v is regular and $q \notin \{i, j, k, \ell\}$, the two circles $C_{p_1q_1}$, $C_{p_2q_2}$ (i.e., C_{kq}, C_{iq}) are distinct. See Figure 5.

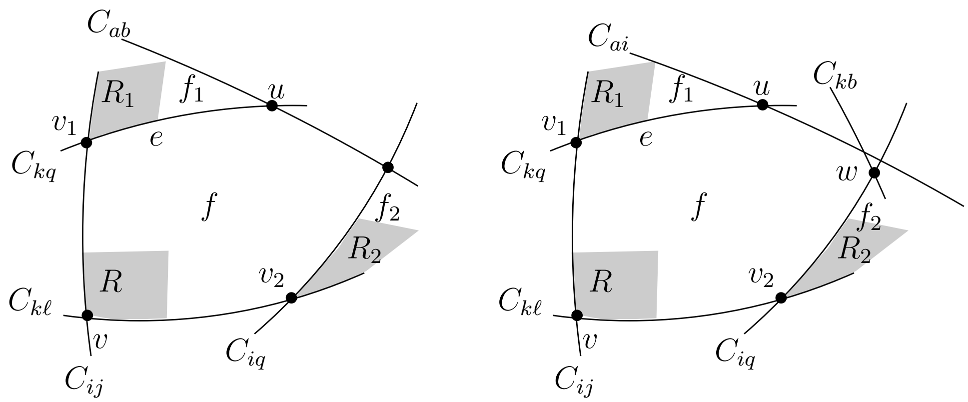

Figure 5: Two scenarios depicting a 1-level border (v, R) of weight 1 that is charged by two 0-level borders $(v_1, R_1), (v_2, R_2)$.

In this special scenario we have two distinct permutation faces f_1 and f_2, where f_1 is the face associated with (v_1, R_1) and f_2 is the face associated with (v_2, R_2).

There are two possible subcases: Assume first that the face f associated with (v, R) is just the quadrangle bounded by C_{ij}, $C_{k\ell}$, C_{iq} and C_{kq}. In this case the fourth vertex of f, formed by intersection of C_{iq} and C_{kq}, is degenerate. Since each degenerate vertex is incident to at most four faces, the number of 1-level borders falling into this subcase is $O(n^3)$.

Suppose then that f has additional edges and vertices. Consider, for example, the vertex u which is the other endpoint (other than v_1) of the edge e of f lying on C_{kq}. Let C_{ab} denote the other circle incident to u. Assume with no loss of generality that v lies in the hemisphere C_{ab}^+. We may also assume that neither a nor b is in $\{k, q\}$, for otherwise u is a degenerate vertex, so we can argue similarly to the previous subcase.

Suppose first that neither a nor b is equal to i. Then removing K_i keeps u as a vertex of $\mathcal{A}(\mathcal{K} \setminus \{K_i\})$. The edge e extends at its other end into a longer *popular* edge (it bounds on one side an extension of $f \cup f_2$ and on the other side an extension of f_1, both of which are now permutation faces; see Figure 5 (left)), so (u, C_{ab}^+) is a 1-level edge border. We charge the 1-level border (v, R) to (u, C_{ab}^+). By construction, such an edge border is charged only once, as is easily checked.

The number of 1-level edge borders can be bounded using the Clarkson-Shor analysis technique [4], similar to the way it was used in the proof of Theorem 3.3. That is, since each 1-level edge border is defined by at most four sets of $\mathcal{K}$ and becomes a 0-level edge border when we remove (at

least) one set from $\mathcal{K}$, the number of 1-level edge borders is $O(\mathbf{E}\{E_0(\mathcal{K}')\})$, where $\mathcal{K}'$ is a random sample of $n/2$ sets of $\mathcal{K}$. Hence, the analysis in the preceding subsection implies that the number of 1-level edge borders in $\mathcal{A}(\mathcal{K})$ is $O(n^3 \log n)$, and therefore the same bound holds for the number of 1-level borders (v, R) under consideration.

We are therefore left with the situation where, say, $b = i$. Applying a fully symmetric argument to the edge of f lying on C_{iq}, we conclude that the only problematic case is where f is at least pentagonal, with five consecutive vertices u, v_1, v, v_2, w, so that u is incident to C_{ai} and C_{kq}, v_1 is incident to C_{kq} and C_{ij}, v is incident to C_{ij} and $C_{k\ell}$, v_2 is incident to $C_{k\ell}$ and C_{iq}, and w is incident to C_{iq} and C_{kb}; here a and b are two indices, neither of which belongs to $\{i, j, k, \ell, q\}$; a and b may be equal. See Figure 5 (right).

Let $\mathcal{A}_i$ be the arrangement of the $n-1$ great circles of the form C_{ir} or C_{ri}, for $r \neq i$. Let f_0 be the face of $\mathcal{A}_i$ containing f. By assumption, the boundary of f touches three distinct boundary edges of f_0. We charge the 1-level border (v, R) to the triple (f_0, e_0, e_1), where $e_0 \subset C_{ij}$ and $e_1 \subset C_{iq}$ are the two boundary edges of f_0 which contain the respective edges of ∂f. To complete the proof of Lemma 3.4, we need the following two lemmas.

Lemma 3.6. *Let $1 \leq i \leq n$, and let $\mathcal{A}_i$ be the arrangement of the $n-1$ great circles C_{ir} or C_{ri}, for $r \neq i$. Let f_0 be a face in $\mathcal{A}_i$, and let e_0, e_1 be two edges of f_0. Then there exist at most two faces of $\mathcal{A}$ which are contained in f_0 and are bounded by a portion of e_0, by a portion of e_1, and by a portion of some other edge of f_0.*

Proof. The edges e_0 and e_1 partition ∂f_0 into up to four connected portions, e_0, γ^-, e_1, γ^+. We claim that there can be at most one face f of $\mathcal{A}$ which is contained in f_0 and which is bounded by a portion of e_0, a portion of e_1, and a portion of γ^+. A symmetric claim holds if we replace γ^+ by γ^-, and the lemma follows. The latter claim follows by observing that the existence of two distinct faces f_1, f_2 of $\mathcal{A}$ contained in f_0 and touching e_0, e_1 and γ^+ would lead to an impossible planar drawing of $K_{3,3}$, as illustrated in Figure 6. See, e.g., [6] for a similar argument. $\square$

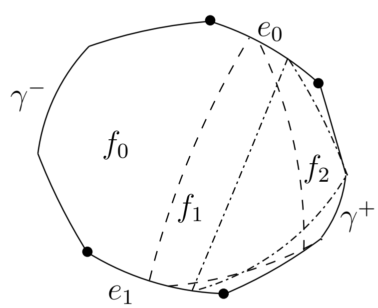

Figure 6: A face f_0 of $\mathcal{A}_i$ cannot contain two distinct faces f_1, f_2 of $\mathcal{A}(\mathcal{K})$ that touch e_0, e_1 and γ^+.

Lemma 3.7. *The number of triples (f_0, e_0, e_1), where f_0 is a face in $\mathcal{A}_i$, as defined in Lemma 3.6, and e_0, e_1 are two edges of f_0, summed over all i, is $O(n^3)$.*

Proof. This follows from the well known result that the sum of the squares of the face complexities in an arrangement of n lines in the plane is $O(n^2)$; see, e.g., [16]. The same analysis applies to an arrangement of great circles on the unit sphere. Summing this bound over all i, the lemma follows. $\square$

Lemma 3.6 implies that any triple (f_0, e_0, e_1), as above, is charged by at most four 1-level borders (v, R). Indeed, the triple determines at most two possible faces f of $\mathcal{A}$, and the edge e_0 determines a unique edge of f with v as one of its endpoints. By Lemma 3.7, the overall number of charged triples (f_0, e_0, e_1) is $O(n^3)$, so the overall number of 1-level borders (v, R) falling into the last subcase is $O(n^3)$. This completes the proof of Lemma 3.4. $\square$

4 Geometric Permutations in Higher Dimensions

In this section we generalize Theorem 3.5 by showing that the number of geometric permutations induced by any collection $\mathcal{K} = \{K_1, \ldots, K_n\}$ of n pairwise disjoint convex sets in $\mathbb{R}^d$ is $O(n^{2d-3} \log n)$, for any $d \geq 3$.

Setup. The basic setup is similar to that in three dimensions, but we repeat it here for the sake of readability. Specifically, we may assume, using the same reasoning as before, that the sets of $\mathcal{K}$ are compact (in addition to being pairwise disjoint and convex). For each $1 \leq i < j \leq n$ we fix some hyperplane h_{ij} which strictly separates K_i and K_j. We orient h_{ij} so that K_i lies in the negative open halfspace h_{ij}^- that it bounds, and K_j lies in the positive open halfspace h_{ij}^+. We represent directions of lines in $\mathbb{R}^d$ by points on the unit $(d-1)$-sphere $\mathbb{S}^{d-1}$. We may assume that the separating hyperplanes h_{ij} are in *general position*, so that every d of them intersect in a unique point, and no $d+1$ of them have a point in common.

Each separating hyperplane h_{ij} induces a great $(d-2)$-sphere C_{ij} on $\mathbb{S}^{d-1}$, which is the locus of the directions of all lines parallel to h_{ij}. C_{ij} partitions $\mathbb{S}^{d-1}$ into two open hemispheres C_{ij}^+, C_{ij}^-, so that C_{ij}^+ (resp., C_{ij}^-), consists of the directions of lines which cross h_{ij} from h_{ij}^- to h_{ij}^+ (resp., from h_{ij}^+ to h_{ij}^-). Any oriented common transversal line of K_i and K_j visits K_j after (resp., before) K_i if and only if its direction lies in C_{ij}^+ (resp., in C_{ij}^-).

Put $\mathcal{C}(\mathcal{K}) = \{C_{ij} \mid 1 \leq i < j \leq n\}$, and consider the arrangement $\mathcal{A}(\mathcal{K})$ of these $\binom{n}{2}$ $(d-2)$-spheres on $\mathbb{S}^{d-1}$. It partitions $\mathbb{S}^{d-1}$ into relatively open cells of dimensions $0, 1, \ldots, d-1$; we refer to an s-dimensional cell of $\mathcal{A}(\mathcal{K})$ simply as an s-cell. The assumption that the hyperplanes h_{ij} are in general position implies that the $(d-2)$-spheres of $\mathcal{C}(\mathcal{K})$ are also in general position, in the sense that the intersection of any s distinct spheres of $\mathcal{C}(\mathcal{K})$, for $1 \leq s \leq d-1$, is a $(d-s-1)$-sphere, and the intersection of any d distinct spheres of $\mathcal{C}(\mathcal{K})$ is empty. Each $(d-1)$-cell f of $\mathcal{A}(\mathcal{K})$ induces a relation $\prec_f$ on $\mathcal{K}$, in which $K_i \prec_f K_j$ (resp., $K_j \prec_f K_i$) if $f \subseteq C_{ij}^+$ (resp., $f \subseteq C_{ij}^-$). The direction of each oriented line transversal λ of $\mathcal{K}$ belongs to the unique $(d-1)$-cell f of $\mathcal{A}(\mathcal{K})$ whose relation $\prec_f$ coincides with the linear order in which λ visits the sets of $\mathcal{K}$. In particular, as noted by Wenger [19], the number of geometric permutations is bounded by the number of $(d-1)$-cells of $\mathcal{A}(\mathcal{K})$, which is $O(n^{2d-2})$.

We call a $(d-1)$-cell f of $\mathcal{A}(\mathcal{K})$ a *permutation cell* if there is at least one line transversal of $\mathcal{K}$ whose direction belongs to f. As in the three-dimensional case, we improve the above bound by showing that the number of permutation cells in $\mathcal{A}(\mathcal{K})$ is $O(n^{2d-3} \log n)$, which also bounds the number of geometric permutations induced by $\mathcal{K}$.

We refer to 0-cells in $\mathcal{A}(\mathcal{K})$ as *vertices*, and to 1-cells as *edges*. We say that a vertex v of $\mathcal{A}(\mathcal{K})$ is *regular* if the $d-1$ $(d-2)$-spheres of $\mathcal{C}(\mathcal{K})$ that are incident to v are defined by $2d-2$ distinct sets of $\mathcal{K}$; otherwise v is a *degenerate* vertex. Clearly, the number of degenerate vertices is $O(n^{2d-3})$, so it suffices to bound the number of regular vertices of permutation cells.

As in the three-dimensional case, we will also consider subcollections $\mathcal{K}'$ of $\mathcal{K}$, typically obtained

by removing one set, say K_q, from $\mathcal{K}$. Doing so eliminates all separating hyperplanes h_{iq}, h_{qi}, as well as all the corresponding $(d-2)$-spheres C_{iq}, C_{qi}, and $\mathcal{A}(\mathcal{K}')$ is constructed only from the remaining spheres. In particular, a vertex[1] v of the intersection $C_{i_1j_1} \cap C_{i_2j_2} \cap \cdots \cap C_{i_{d-1}j_{d-1}}$ of $\mathcal{A}(\mathcal{K})$ remains a vertex of $\mathcal{A}(\mathcal{K}')$ if and only if $q \notin \{i_1, j_1, \ldots, i_{d-1}, j_{d-1}\}$. A cell of $\mathcal{A}(\mathcal{K}')$, of any dimension $s \geq 1$, may contain several cells of $\mathcal{A}(\mathcal{K})$. As before, if f' is a $(d-1)$-cell of $\mathcal{A}(\mathcal{K}')$ which contains a permutation cell f of $\mathcal{A}(\mathcal{K})$ then f' is a permutation cell in $\mathcal{A}(\mathcal{K}')$; the permutation that it induces is the permutation of f with K_q removed.

Each s-cell f of $\mathcal{A}(\mathcal{K})$ is incident to 2^{d-s-1} $(d-1)$-cells of $\mathcal{A}(\mathcal{K})$. If all these cells are permutation cells, f is called *popular*. In particular, a popular vertex is incident to 2^{d-1} permutation cells, a popular edge is incident to 2^{d-2} permutation cells, and a popular $(d-1)$-cell is a permutation cell.

Overview. We show that the number of popular vertices is $O(n^{2d-3})$ by a straightforward generalization of the analysis in Section 3.1. The analysis then proceeds by applying, for each $1 \leq s \leq d-1$, a charging scheme, which expresses the number of popular s-cells in terms of the number of popular $(s-1)$-cells (and degenerate vertices). A naive charging scheme produces a recurrence whose solution incurs an additional logarithmic factor for each s, resulting in the weaker bound $g_d(n) = O(n^{2d-3} \log^{d-1} n)$. A more careful analysis, as in the three-dimensional case, leads to refined recurrences, whose solution yields the improved bound $g_d(n) = O(n^{2d-3} \log n)$. (We lose a logarithmic factor only when passing from vertices to edges, as in the three-dimensional case.)

4.1 The number of popular vertices

For a regular vertex $v \in \bigcap_{q=1}^{d-1} C_{i_qj_q}$ of $\mathcal{A}(\mathcal{K})$, we denote by $\mathcal{K}_v$ the collection $\{K_{i_q}, K_{j_q} \mid 1 \leq q \leq d-1\}$ of the $2d-2$ sets defining v.

Lemma 4.1. *Let $v \in \bigcap_{q=1}^{d-1} C_{i_qj_q}$ be a regular popular vertex of $\mathcal{A}(\mathcal{K})$.*
(i) Each pair of sets $K_a \in \mathcal{K}_v$ and $K_b \in \mathcal{K} \setminus \mathcal{K}_v$ appear in the same order in all the 2^{d-1} permutations induced by the $(d-1)$-cells incident to v.
(ii) The elements of each pair $K_{i_q}, K_{j_q} \in \mathcal{K}_v$, for $1 \leq q \leq d-1$, are consecutive in all these 2^{d-1} permutations.

Proof. Each pair of distinct $(d-1)$-cells f, g incident to v are separated by at most $d-1$ $(d-2)$-spheres from $\{C_{i_1j_1}, \ldots, C_{i_{d-1}j_{d-1}}\}$, and only by these spheres. Hence the orders $\prec_f$ and $\prec_g$ may disagree only over the pairs (K_{i_q}, K_{j_q}), for $1 \leq q \leq d-1$. As in the proof of Lemma 3.1, this is easily seen to imply both parts of the lemma. $\square$

Lemma 4.2. *Let $v \in \bigcap_{q=1}^{d-1} C_{i_qj_q}$ be a regular popular vertex in $\mathcal{A}(\mathcal{K})$. Then the line $\lambda_v = \bigcap_{q=1}^{d-1} h_{i_qj_q}$ stabs all the $n-2d+2$ sets in $\mathcal{K} \setminus \mathcal{K}_v$, and misses all the $2d-2$ sets in $\mathcal{K}_v$.*

Proof. By definition, λ_v misses every set $K \in \mathcal{K}_v$, because it is contained in a hyperplane separating K from another set in $\mathcal{K}_v$. Hence, it suffices to show that λ_v is a transversal of $\mathcal{K} \setminus \mathcal{K}_v$.

To show this, we fix a set $K_a \in \mathcal{K} \setminus \mathcal{K}_v$ and show that each of the 2^{d-1} wedges determined by $\{h_{i_qj_q} \mid 1 \leq q \leq d-1\}$ meets K_a. Each of these wedges is the intersection of $d-1$ halfspaces, where the q-th halfspace is either $h^+_{i_qj_q}$ or $h^-_{i_qj_q}$, for $q = 1, \ldots, d-1$. All these wedges have λ_q on their boundary, and the convexity of K_a then implies, exactly as in the three-dimensional case, that λ_v intersects K_a.

[1] As in the three-dimensional case, the intersection consists of two antipodal points, so there are two choices for v.

For specificity, we show that K_a intersects the wedge $\bigcap_{q=1}^{d-1} h_{i_q j_q}^+$; the proof for the other wedges is essentially the same. Lemma 4.1 implies that K_a lies at the same position in each of the 2^{d-1} permutations induced by the cells incident to v. For each index q, if K_{i_q}, K_{j_q} appear before K_a (resp., after K_a) in all permutations induced by the cells incident to v, put $C_q = C_{i_q j_q}^+$ (resp., $C_q = C_{i_q j_q}^-$).

Let f be the cell incident to v and contained in $\bigcap_{q=1}^{d-1} C_q$, and let μ_f be a transversal line stabbing $\mathcal{K}$ in the order $\prec_f$ (so its direction lies in f). By the choice of f and by our assumption, we have either $K_{i_q} \prec_f K_{j_q} \prec_f K_a$, or $K_a \prec_f K_{j_q} \prec_f K_{i_q}$. This implies in the former case that μ_f visits K_a after crossing $h_{i_q j_q}$ from $h_{i_q j_q}^-$ (the side containing K_{i_q}) to $h_{i_q j_q}^+$ (the side containing K_{j_q}). In the latter case, μ_f first visits K_a and then crosses $h_{i_q j_q}$ from $h_{i_q j_q}^+$ to $h_{i_q j_q}^-$. Thus, in either case, the segment $\lambda_f \cap K_a$ lies in $h_{i_q j_q}^+$, and this holds for every $1 \le q \le d-1$. Hence $\lambda_f \cap K_a \subset \bigcap_{q=1}^{d-1} h_{i_q j_q}^+$, and the claim follows. $\square$

Theorem 4.3. *Let $\mathcal{K}$ be a collection of n pairwise disjoint compact convex sets in $\mathbb{R}^d$. Then the number of popular vertices in $\mathcal{A}(\mathcal{K})$ is $O(n^{2d-3})$.*

Proof. As in the three-dimensional case, it is easily checked that a popular vertex must be regular. Let $v \in \bigcap_{q=1}^{d-1} C_{i_q j_q}$ be a (regular) popular vertex in $\mathcal{A}(\mathcal{K})$, and let $\lambda_v = \bigcap_{q=1}^{d-1} h_{i_q j_q}$ be the intersection line of the corresponding hyperplanes. Consider the plane $H = \bigcap_{q=1}^{d-2} h_{i_q j_q}$, put $K_a^* = K_a \cap H$, for each index $a \notin \{i_q, j_q \mid 1 \le q \le d-2\}$, and denote by $\mathcal{K}^*$ the collection of these $n - 2d + 4$ planar cross-sections. Clearly, all sets in $\mathcal{K}^*$ are pairwise disjoint, compact, and convex.

By Lemma 4.2, λ_v lies in H, stabs all the sets in $\mathcal{K}^* \setminus \{K_{i_{d-1}}^*, K_{j_{d-1}}^*\}$, and misses the two sets $K_{i_{d-1}}^*, K_{j_{d-1}}^*$. As in Theorem 3.3, we charge λ_v to an extremal line $\mu = \mu_v$ within H which is tangent to two sets of $\mathcal{K}^*$, and misses only the sets among $K_{i_{d-1}}^*, K_{j_{d-1}}^*$ that it does not touch. As in the preceding analysis, each extremal line μ of this kind is charged at most twice. Applying the Clarkson-Shor analysis [4], similarly to Theorem 3.3, the number of lines μ, charged within H, is $O(n)$. Summing over all possible choices of the 2-planes H, namely over all choices of $d-2$ of the hyperplanes h_{ij}, the number of lines λ_v, and thus the number of popular vertices, is $O(n \cdot n^{2d-4}) = O(n^{2d-3})$. $\square$

4.2 The number of permutation cells

We next generalize the analysis of Section 3.3 to higher dimensions. We first extend the notion of borders. Let v be a vertex of $\mathcal{A}(\mathcal{K})$, so that $v \in \bigcap_{1 \le q \le d-1} C_{i_q j_q}$. For any subset J of $\{1, \ldots, d-1\}$, let $R \subseteq \mathbb{S}^{d-1}$ be a connected component of $\mathbb{S}^{d-1} \setminus \bigcup_{q \in J} C_{i_q j_q}$. Equivalently, it is the intersection of $|J|$ hemispheres, where the q-th hemisphere, for $q \in J$, is either $C_{i_q j_q}^+$ or $C_{i_q j_q}^-$. Note that there are $2^{|J|}$ such regions (for any fixed J). We call (v, R) an *s-border*, where $s = |J|$. Given v and R, for $s \ge 1$, there is a unique s-dimensional cell f of $\mathcal{A}(\mathcal{K})$ which is incident to v and is contained in the interior of R. This cell f lies in the intersection of the interior of R with $\bigcap_{q \in J^c} C_{i_q j_q}$, where $J^c = \{1, \ldots, d-1\} \setminus J$. We refer to f as the s-cell of $\mathcal{A}(\mathcal{K})$ associated with (v, R). For $s = 0$ we define the s-cell of $\mathcal{A}(\mathcal{K})$ associated with (v, R) to be v itself, and for $s = d-1$ we define the s-cell of $\mathcal{A}(\mathcal{K})$ associated with (v, R) to be the unique $(d-1)$-cell incident to v and contained in R. The reader is invited to check that, for $d = 3$, a 0-border, in the new definition, is a vertex of $\mathcal{A}(\mathcal{K})$, a 1-border is an edge border, and a 2-border is what we simply called a border.

Let (v, R) be an s-border and let f be the s-cell associated with (v, R). If f is a popular cell, we say that (v, R) is a 0-*level* s-*border* of $\mathcal{A}(\mathcal{K})$. An s-border (v, R) is a 1-*level* s-*border* in $\mathcal{A}(\mathcal{K})$ if it is not a 0-level s-border, but becomes such a border after removing from $\mathcal{K}$ some single set K. In this case we say that K is *in conflict* with (v, R). As in the three-dimensional case, K need not be unique.

For each $t = 0, 1$ and $0 \le s \le d-1$, let $N_t^{(s)}(\mathcal{K})$ be the number of t-level s-borders in $\mathcal{A}(\mathcal{K})$, and let $N_t^{(s)}(n)$ denote the maximum value of $N_t^{(s)}(\mathcal{K})$, over all collections $\mathcal{K}$ of n pairwise disjoint compact convex sets in $\mathbb{R}^d$.

Note that $N_0^{(0)}(\mathcal{K})$ is the number of popular vertices in $\mathcal{A}(\mathcal{K})$, so we have $N_0^{(0)}(n) = O(n^{2d-3})$. The term $N_0^{(d-1)}(\mathcal{K})$ counts the overall number of vertices incident to permutation cells, where each vertex is counted once for each permutation cell incident to it. Assuming $n \ge d$, each permutation cell in $\mathcal{A}(\mathcal{K})$ is incident to at least one vertex (and each vertex is incident to at most 2^{d-1} permutation cells). Thus, the number of geometric permutations of $\mathcal{K}$ is at most $N_0^{(d-1)}(\mathcal{K})$.

For each $1 \le s \le d-1$, we apply a charging scheme, which results in a recurrence which expresses $N_0^{(s)}(\mathcal{K})$ in terms of $N_0^{(s-1)}(\mathcal{K})$ and $N_1^{(s)}(\mathcal{K})$.

Fix $1 \le s \le d-1$. Let (v, R) be a 0-level s-border in $\mathcal{A}(\mathcal{K})$, and let f be the popular s-cell associated with (v, R). Let $C_{i_1j_1}, C_{i_2j_2}, \ldots, C_{i_{d-1}j_{d-1}}$ be the $d-1$ $(d-2)$-spheres of $\mathcal{C}(\mathcal{K})$ incident to v, and assume, with no loss of generality, that $R = \bigcap_{q=1}^{s} C_{i_qj_q}^+$. Moreover, we may assume that v is regular, since the number of s-borders incident to degenerate vertices is clearly $O(n^{2d-3})$. To simplify the notation, we refer to borders incident to a regular (resp., degenerate) vertex as *regular borders* (resp., *degenerate borders*).

For each $1 \le q \le s$ there exists a unique edge e_q^+ of f which is incident to v and not contained in $C_{i_qj_q}$. Indeed, by construction, f lies in the intersection s-sphere $\bigcap_{q=s+1}^{d-1} C_{i_qj_q}$ (for $s = d-1$, this is the entire $\mathbb{S}^{d-1}$), and each edge of f incident to v is formed by further intersecting this sphere with $s-1$ additional spheres from $C_{i_1j_1}, \ldots, C_{i_sj_s}$. The claim follows since only one side of the resulting intersection circle lies (near v) in the closure of R. Let e_q^- denote the other edge of $\mathcal{A}(\mathcal{K})$ which is incident to v and lies on the same intersection circle γ as e_q^+, so e_q^- emanates from v away from R. Let v_q denote the other endpoint of e_q^-, and let g_q denote the (unique) $(s-1)$-cell which bounds f, lies in $C_{i_qj_q} \cap \left(\bigcap_{t=s+1}^{d-1} C_{i_tj_t}\right)$, is incident to v and is contained in $R_q = \bigcap_{1 \le t \le s,\, t \ne q} C_{i_tj_t}^+$. Also, g_q is the $(s-1)$-cell associated with (v, R_q). See Figure 7. There are two possible cases:

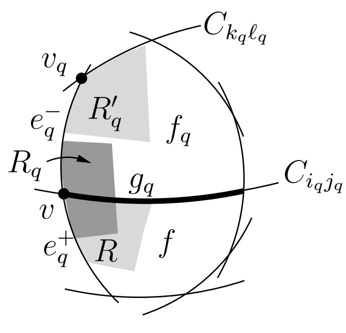

Figure 7: Charging a 0-level s-border along the edge e_q.

(i) If one of the $(s-1)$-cells $g_1, g_2, \ldots, g_s$, say g_s, is popular, we charge (v, R) to the 0-level

$(s-1)$-border (v, R_s), noting, as above, that g_s is the $(s-1)$-cell associated with this border. By construction, each 0-level $(s-1)$-border (v, R) is charged at most $2(d-s)$ times in this manner, once from each s-border associated with an s-cell which is bounded by the $(s-1)$-cell associated with this border (there are $d-s$ choices for the great sphere $C_{i_t j_t}$ that participates in the definition of (v, R) but is absent in the s-border, and two choices of the corresponding hemisphere $C^+_{i_t j_t}, C^-_{i_t j_t}$). Hence, the number of 0-level s-borders falling into subcase (i) is $O\left(N_0^{(s-1)}(\mathcal{K})\right)$.

(ii) None of the $(s-1)$-cells $g_1, g_2, \ldots, g_s$ is popular. For each $1 \le q \le s$, let $C_{k_q \ell_q}$ be the additional great sphere incident to v_q, and suppose, for specificity, that $v \in C^+_{k_q \ell_q}$. The vertex v_q participates in the 1-level s-border (v_q, R'_q), where $R'_q = C^+_{k_q \ell_q} \cap \left(\bigcap_{1 \le t \le s,\, t \ne q} C^+_{i_t j_t}\right)$.

Since g_q is not popular, (v_q, R'_q) is not a 0-level s-border. Let f_q be the s-cell associated with (v_q, R'_q). Clearly, at least one of i_q, j_q does not belong to $\{k_q, \ell_q\}$; say it is i_q. Thus, and since v is regular, removing K_{i_q} keeps v_q (and hence (v_q, R'_q)) intact, and makes f and f_q fuse into a larger s-cell f' containing both of them. Clearly, f' is the cell associated with (v_q, R'_q) in $\mathcal{A}(\mathcal{K} \setminus \{K_{i_q}\})$, and it is popular there because $f \subset f'$ was popular in $\mathcal{A}(\mathcal{K})$. We say that the borders (v, R), (v_q, R'_q) are *neighbors* in $\mathcal{A}(\mathcal{K})$.

We then charge (v, R) to its s neighboring 1-level s-borders (v_q, R'_q), for $q = 1, \ldots, s$. Note that each 1-level s-border (v, R) is charged at most s times, once along each of the s edges, incident to v, of the s-cell associated with it. We thus obtain the following recurrence.

$$N_0^{(s)}(\mathcal{K}) \le N_1^{(s)}(\mathcal{K}) + O\left(N_0^{(s-1)}(\mathcal{K}) + n^{2d-3}\right), \tag{5}$$

where the first term in the right hand side bounds the number of 0-level s-borders falling into case (ii), and the second term bounds the number of the remaining 0-level s-borders.

Similarly to the three-dimensional case, we combine the system (5) of recurrences with the analysis technique of Tagansky, and solve the resulting recurrences to obtain a slightly inferior bound (involving a larger polylogarithmic factor). We then refine the recurrences, using a more careful analysis, similar to the one in Section 3, and thereby obtain the improved bound $O(n^{2d-3} \log n)$.

Applying Tagansky's technique: The simpler variant. We prove that $N_0^{(s)}(n) = O(n^{2d-3} \log^s n)$ by induction on s. For the base case $s = 0$, we have $N_0^{(0)}(n) = O(n^{2d-3})$ by Theorem 4.3. Consider a fixed $s \ge 1$ and assume that the bound holds for $s-1$, so (5) becomes

$$N_0^{(s)}(\mathcal{K}) \le N_1^{(s)}(\mathcal{K}) + O(n^{2d-3} \log^{s-1} n). \tag{6}$$

Let $\mathcal{R}$ be a random sample of $n-1$ sets of $\mathcal{K}$, obtained by removing a random set K from $\mathcal{K}$. The expected number of 0-level popular s-borders in $\mathcal{A}(\mathcal{R})$ satisfies

$$\mathbf{E}\{N_0^{(s)}(\mathcal{R})\} \ge \frac{n-2d+2}{n} N_0^{(s)}(\mathcal{K}) + \frac{1}{n} N_1^{(s)}(\mathcal{K}). \tag{7}$$

This follows since a 0-level s-border (v, R) (with v regular) survives after removing K if and only if $K \notin \mathcal{K}_v$, and a 1-level s-border becomes a 0-level s-border if and only if it is in conflict with K. Combining this inequality with (6), we get

$$\begin{aligned} \frac{1}{n} N_0^{(s)}(\mathcal{K}) &\le \frac{1}{n} N_1^{(s)}(\mathcal{K}) + O(n^{2d-4} \log^{s-1} n) \le \\ &\mathbf{E}\left\{N_0^{(s)}(\mathcal{R})\right\} - \frac{n-2d+2}{n} N_0^{(s)}(\mathcal{K}) + O(n^{2d-4} \log^{s-1} n), \end{aligned}$$

or

$$\frac{n-2d+3}{n} N_0^{(s)}(\mathcal{K}) \le \mathbf{E}\left\{ N_0^{(s)}(\mathcal{R}) \right\} + O(n^{2d-4}\log^{s-1} n).$$

Replacing $N_0^{(s)}(\mathcal{K})$ and $N_0^{(s)}(\mathcal{R})$ by their respective maximum possible values $N_0^{(s)}(n)$ and $N_0^{(s)}(n-1)$, we get the recurrence

$$\frac{n-2d+3}{n} N_0^{(s)}(n) \le N_0^{(s)}(n-1) + O(n^{2d-4}\log^{s-1} n),$$

whose solution is easily seen to be $N_0^{(s)}(n) = O(n^{2d-3}\log^{s} n)$. This establishes the induction step and thus proves the asserted bound. In particular, we have so far

$$g_d(n) = O(n^{2d-3}\log^{d-1} n).$$

Improved bounds for $s \ge 2$. As promised, we next refine the analysis, and show that

$$N_0^{(s)}(\mathcal{K}) = O(n^{2d-3}\log n), \tag{8}$$

for any $1 \le s \le d-1$, by establishing a sharper variant of (5).

As in the three-dimensional case, the weakness of the preceding analysis lies in the random sampling inequality (7), or, more precisely, in the term $N_1^{(s)}(\mathcal{K})/n$ thereof.

Specifically, if a 1-level s-border (v, R) is in conflict with $w > 1$ sets of $\mathcal{K}$ then removing any one of these sets will make (v, R) a 0-level s-border, so the probability of this to happen is w/n, which is significantly larger than the bound $1/n$ used in (7). As above, we refer to w as the *weight* of (v, R). We can therefore modify the definition of $N_1^{(s)}(\mathcal{K})$ so a border of weight w is counted w times. The preceeding discussion ensures that (7) still holds in the new setting.

We proceed to prove 8 by induction on s. The base case $s = 1$ has already been analyzed, and we have shown that $N_0^{(1)}(n) = O(n^{2d-3}\log n)$. Fix $2 \le s \le d-1$, and suppose that we have already proved that $N_0^{(s')}(\mathcal{K}) = O(n^{2d-3}\log n)$, for all $1 \le s' < s$.

The following lemma generalizes Lemma 3.4 to arbitrary dimension $d \ge 4$.

Lemma 4.4. *(i) The number of 1-level 2-borders, having weight 1 and charged by two 0-level neighboring 2-borders, is* $O(N_1^{(1)}(\mathcal{K}) + n^{2d-3})$.
(ii) For $s \ge 3$, there are no 1-level s-borders incident to a regular vertex, having weight 1, and charged by s 0-level neighboring s-borders.

Proof of Lemma 4.4. The proof of (i) is very similar to the proof of Lemma 3.4, and will be briefly presented later, after we prove (ii).

So we assume that $s \ge 3$. Let (v, R) be a 1-level s-border which has weight 1 and is charged by s 0-level neighboring s-borders, so that v is regular. Let $C_{i_1j_1}, C_{i_2j_2}, \ldots, C_{i_{d-1}j_{d-1}}$ be the $(d-2)$-spheres incident to v. Without loss of generality, assume that $R = \bigcap_{q=1}^{s} C^+_{i_qj_q}$. Let f be the s-cell associated with (v, R), and let $e_1, \ldots, e_s$ be the s edges of f incident to v, so that, for each $k = 1, \ldots, s$, the edge e_k lies on the circle $\bigcap_{1\le q\le d-1, q\ne k} C_{i_qj_q}$. For $k = 1, \ldots, s$, let v_k denote the other endpoint of e_k, and let $C_{a_kb_k}$ denote the (unique) great sphere incident to v_k and not containing e_k. Assume, without loss of generality, that v lies in $C^-_{a_kb_k}$, and put $R_k = C^+_{a_kb_k} \cap \bigcap_{1\le q\le s, q\ne k} C^+_{i_qj_q}$. By construction, the s s-borders (v_k, R_k), for $k = 1, \ldots, s$, are precisely those that charge (v, R), so they are all regular 0-level s-borders.

Note that (v, R) is in conflict with each of the sets $K_{a_1}, K_{b_1}, \ldots, K_{a_s}, K_{b_s}$ for which the corresponding index a_k or b_k is not one of $i_1, j_1, \ldots, i_{d-1}, j_{d-1}$. Indeed, removing such a set K_{a_k}, say, eliminates the sphere $C_{a_k b_k}$ and thereby exposes v to the extended 2^{d-1-s} permutation cells that surround v_k, so that they are all now contained in R, so (v, R) becomes a 0-level s-border. However, since the weight of (v, R) is 1, only one of these sets, call it K_b, can be in conflict with (v, R) (so $b \notin \{i_1, j_1, \ldots, i_{d-1}, j_{d-1}\}$). This, and the fact that each of the v_k's is regular, is easily seen to imply the following property: For each k, one of a_k, b_k, say a_k, belongs to $\{i_k, j_k\}$, and the other index b_k is b.

Fix a pair of distinct vertices v_k, v_ℓ, and denote by Π_k (resp., Π_ℓ) the collection of the 2^{d-1-s} permutations induced by the permutation cells that surround v_k (resp., v_ℓ) and are contained in R_k (resp., R_ℓ). Any pair of permutations in Π_k differ from each other only by swaps of some of the pairs (i_q, j_q), for $q = s+1, \ldots, d-1$. Hence the indices of each of these pairs appear consecutively in any of these permutations, and the locations of these pairs are fixed for all permutations. The set K_b appears, somewhere in between these pairs, in a fixed location in all permutations. A similar property holds for the permutations in Π_ℓ.

Fix a permutation $\pi \in \Pi_k$. It has a "twin" permutation π' in Π_ℓ, in which the order of the two indices in each of the pairs (i_q, j_q), for $q = s+1, \ldots, d-1$, is the same as their order in π. To gain more insight into the structure of π and π', let φ and φ' denote, respectively, the permutation cells of $\mathcal{A}(\mathcal{K})$ in which π and π' are generated. We can get from φ to φ' by first crossing $C_{i_k b}$ into a corresponding $(d-1)$-cell φ_0 surrounding f and then cross $C_{i_\ell b}$ into φ'. This means that $\prec_\varphi$ and $\prec_{\varphi'}$ (i.e., π and π') are obtained from each other by first swapping K_b with K_{i_k} and then by swapping K_b with K_{i_ℓ}. As is easily checked, this implies that K_{i_k} and K_{i_ℓ} must be adjacent in π and in π'. This however cannot hold for *every* pair of distinct indices in $\{i_1, \ldots, i_s\}$ if $s \geq 3$. This contradiction shows that for $s \geq 3$ there are no 1-level s-borders which satisfy the assumptions in the lemma. This completes the proof of part (ii).

We now consider the case $s = 2$, which, as noted above, can be handled in a manner that is very similar to the analysis in Lemma 3.4. Specifically, let (v, R) be a regular 1-level 2-border of weight 1 which is charged by two 0-level 2-borders (v_1, R_1), (v_2, R_2). (The number of degenerate 1-level 2-borders is $O(n^{2d-3})$.) As in the proof of part (i), we may assume that both v_1 and v_2 are regular (for otherwise they would not charge (v, R)). Let $C_{i_1 j_1}, C_{i_2 j_2}, \ldots, C_{i_{d-1} j_{d-1}} \in \mathcal{C}(\mathcal{K})$ be the $(d-2)$-spheres incident to v, and assume that $R = C^+_{i_1 j_1} \cap C^+_{i_2 j_2}$. Let f be the 2-face associated with (v, R). For $k = 1, 2$, let e_k denote the edge of f incident to v and contained in $C_{i_k j_k}$, and assume that v_k is the other endpoint of e_k. Let $C_{a_k b_k}$ be the (unique) great sphere passing through v_k and not containing e_k.

As in the three-dimensional case, and similar to the preceding analysis, since (v, R) has weight 1, the only case to be considered, up to symmetry, is where $a_1 = i_2$, $a_2 = i_1$, and $b_1 = b_2 = b$, where $b \neq \{i_1, j_1, \ldots, i_{d-1}, j_{d-1}\}$.

The proof now continues as in the three-dimensional case, and we only provide a brief sketch of it. For $k = 1, 2$, we consider the other edge e'_k of f incident to v_k, denote by u_k the other endpoint of e_k, and assume that neither of u_1, u_2 is degenerate. See Figure 8. We then consider the other great sphere $C_{r_k s_k}$ incident to u_k, for $k = 1, 2$, and distinguish between the following two cases:
(a) $i_1 \neq r_1, s_1$ or $i_2 \neq r_2, s_2$. In the former case, removing K_{i_1} leaves u_1 intact and extends e'_1 into a 0-level 1-border; the proof is argued exactly as in the three-dimensional case. The latter case is handled symmetrically, and we conclude that the number of 2-borders of this kind is $O(N_1^{(1)}(\mathcal{K}))$.
(b) $i_1 = r_1$ and $i_2 = r_2$ (or any of the symmetric pairs of equalities). In this case we consider the

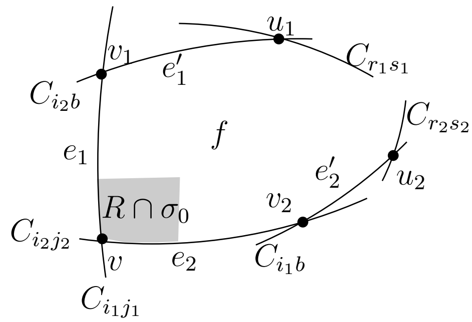

Figure 8: The setup in the proof of Theorem 4.4 for $s = 2$: View within the sphere $\sigma_0 = \bigcap_{q=3}^{d-1} C_{i_qj_q}$.

2-sphere $\sigma_0 = \bigcap_{q=3}^{d-1} C_{i_qj_q}$ which contains f, and construct in it the arrangement $\mathcal{A}_{i_1}^{(\sigma_0)}$, formed by the circles $C_{i_1x} \cap \sigma_0$, for $x \notin \{i_1\} \cup \{i_3, j_3, \ldots, i_{d-1}, j_{d-1}\}$. We note that f is contained in a face f_0 of $\mathcal{A}_{i_1}^{(\sigma_0)}$ and touches its boundary at three distinct edges. This allows us to bound the number of 2-borders under consideration by $O(n^3)$, for a fixed choice of $i_3, j_3, \ldots, i_{d-1}, j_{d-1}$, arguing exactly as in the three-dimensional case. In total, the number of these 2-borders is $O(n^3 \cdot n^{2d-6}) = O(n^{2d-3})$. This completes the proof of the lemma. $\square$

First, for $s = 2$, we bound the quantity $N_1^{(1)}(\mathcal{K})$ using the Clarkson-Shor analysis technique [4], as we did in the proof of Lemma 3.4. That is, since each 1-level 1-border is defined by at most $2d-2$ sets of $\mathcal{K}$ and becomes a 0-level 1-border when we remove (at least) one set from $\mathcal{K}$, the number of 1-level 1-borders is $O\left(\mathbf{E}\{N_0^{(1)}(\mathcal{K}')\}\right)$, where $\mathcal{K}'$ is a random sample of $n/2$ sets of $\mathcal{K}$. Thus, combining this with the bound already established for $s = 1$, we have

$$N_1^{(1)}(\mathcal{K}) = O\left(\mathbf{E}\{N_0^{(1)}(\mathcal{K}')\}\right) = O\left(N_0^{(1)}(n/2)\right) = O(n^{2d-3}\log n). \tag{9}$$

With these preparations, we are now ready to complete the induction step for s.

Let $N_{1,1}^{(s)}(\mathcal{K})$ denote the number of 1-level s-borders having weight 1, and let $N_{1,2}^{(s)}(\mathcal{K})$ denote the number of 1-level s-borders having weight *at least* 2. Since a 1-level s-border of weight w_i contributes to $N_1^{(s)}(\mathcal{K})$ w_i units, we have

$$N_1^{(s)}(\mathcal{K}) \geq N_{1,1}^{(s)}(\mathcal{K}) + 2N_{1,2}^{(s)}(\mathcal{K}). \tag{10}$$

Recall that we charge every 0-level s-border (falling into subcase (ii)) to s neighboring 1-level s-borders. By Lemma 4.4 (and (9)), all but $O(n^{2d-3}\log n)$ 1-level s-borders, that have weight 1, are charged by at most $s-1$ neighboring 0-level s-borders. (This is the situation for $s = 2$; the bound drops to $O(n^{2d-3})$ for $s \geq 3$.) Thus, we obtain the following refinement of 5:

$$sN_0^{(s)}(\mathcal{K}) \leq (s-1)N_{1,1}^{(s)}(\mathcal{K}) + sN_{1,2}^{(s)}(\mathcal{K}) + O(n^{2d-3}\log n). \tag{11}$$

The combination of (10) and (11), and the assumption that $s \geq 2$ (so $s/(s-1) \geq 2$) imply that,

$$\frac{s}{s-1}N_0^{(s)}(\mathcal{K}) \leq N_1^{(s)}(\mathcal{K}) + O(n^{2d-3}\log n).$$

Substituting $t = \frac{s}{s-1} - 1 = \frac{1}{s-1} > 0$ and combining this with (7), we get

$$\frac{1+t}{n} N_0^{(s)}(\mathcal{K}) \le \frac{1}{n} N_1^{(s)}(\mathcal{K}) + O(n^{2d-4}\log n)$$

$$\le \mathbf{E}\left\{N_0^{(s)}(\mathcal{R})\right\} - \frac{n-2d+2}{n} N_0^{(s)}(\mathcal{K}) + O(n^{2d-4}\log n),$$

or

$$\frac{n-2d+3+t}{n} N_0^{(s)}(\mathcal{K}) \le \mathbf{E}\left\{N_0^{(s)}(\mathcal{R})\right\} + O(n^{2d-4}\log n).$$

Thus, as above, we get the following recurrence

$$\frac{n-2d+3+t}{n} N_0^{(s)}(n) \le N_0^{(s)}(n-1) + O(n^{2d-4}\log n),$$

whose solution is easily seen to be

$$N_0^{(s)}(n) = O(n^{2d-3}\log n)$$

(see, e.g., [18, Proposition 3.1]), which readily implies Theorem 4.5. This completes the induction step and thus establishes 8 for all s. We thus obtain the main result of the paper.

Theorem 4.5. *Any collection $\mathcal{K}$ of n pairwise disjoint convex sets in $\mathbb{R}^d$, for any $d \ge 3$, admits at most $O(n^{2d-3}\log n)$ geometric permutations.*

5 Discussion

Although the improvement presented in this paper is significant, especially since no progress was made on the problem during the past 20 years, it is far from satisfactory, since we strongly believe (and tend to conjecture) that the correct upper bounds are close to $O(n^{d-1})$, for any $d \ge 3$. Improving further the bounds is the main open problem left by this study. A modest subgoal is to get rid of the logarithmic factor in our bounds, and show, e.g., that $g_3(n) = O(n^3)$.

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